THE APPLICATION OF THE FUNDAMENTAL LAWS OF ALGEBRA TO THE MULTIPLICATION OF INFINITE SERIES.

BY PROFESSOR FLORIAN CAJORI.

THE present writer has given examples in which an absolutely convergent series is obtained as the result of multiplying two conditionally convergent series together, or of multiplying one conditionally convergent series by a divergent series.*

He has also given an example of two divergent series whose product is absolutely convergent.†

Pringsheim has treated this subject from a more general point of view and by very simple methods has shown that the property in question is typical of certain classes of series.‡

In the present paper it is proposed to establish a class of series with real terms, possessing the property alluded to, but which seems to be distinct from the class given by Pringsheim. Next, we shall consider the validity of the fundamental laws of algebra in the multiplication of infinite series. Then, with aid of our conclusions relating to these laws, we shall point out another method for obtaining divergent series whose product is absolutely convergent. Lastly we shall generalize a theorem of Abel on the multiplication of series.

§ 1.

In the series S_1 and S_2, obtained respectively by removing the parentheses from the series

$$S_1' \equiv \sum_{\nu=0}^{\nu=\infty} (a_{4\nu} - a_{4\nu+1} + a_{4\nu+2} - a_{4\nu+3}),$$

$$S_2' \equiv \sum_{\nu=0}^{\nu=\infty} (b_{4\nu} + b_{4\nu+1} - b_{4\nu+2} - b_{4\nu+3}),$$

wherein the a's and b's are real and positive, let the following conditions be satisfied:

(1) The νth term in S_1 and in S_2 shall be $\equiv \nu^{-r}$, where $\frac{1}{2} < r \equiv 1$, but $\sum a_\nu$ and $\sum b_\nu$ are both divergent.

* *Trans. Amer. Math. Society*, vol. 2, pp. 25–36, January, 1901.
† *Science*, new ser., vol. 14, p. 395 (September 13, 1901)
‡ *Trans. Amer. Math. Society*, vol. 2, pp. 404–412, October, 1901.

(2) $a_{4\nu} = a_{4\nu+3}$, $a_{4\nu+1} = a_{4\nu+3}$, $b_{4\nu} = b_{4\nu+1}$, $b_{4\nu+2} = b_{4\nu+3}$;

the binomials

$$a_{4\nu} - a_{4\nu+1}, \quad b_{4\nu+3} - b_{4\nu},$$

$$a_{4(\nu+1)} - a_{4\nu+1}, \quad b_{4\nu+2} - b_{4(\nu+1)},$$

shall each differ from $\varepsilon_{4\nu+3}$ by less than $k_{4\nu+3}$, where

$$\varepsilon_s = \frac{1}{s(\lg s)^\lambda}, \quad \lambda > 1, \quad \text{and} \quad k_s = \frac{1}{s^{2-\gamma}(\lg s)^\lambda}.$$

The $(4n)$th term of the product of S_1 and S_2 is

$$\sum_{\nu=0}^{\nu=n-1} (-b_{4\nu} \cdot a_{4(n-\nu-1)+3} + b_{4\nu+1} \cdot a_{4(n-\nu-1)+2} + b_{4\nu+2} \cdot a_{4(n-\nu-1)+1} - b_{4\nu+3} \cdot a_{4(n-\nu-1)}),$$

$$= \sum_{\nu=0}^{\nu=n'-1} \{b_{4\nu}(a_{4(n-\nu-1)+2} - a_{4(n-\nu-1)+3}) - b_{4\nu+2}(a_{4(n-\nu-1)} - a_{4(n-\nu-1)+1})\}$$

$$+ \sum_{\nu=n'}^{\nu=n-1} \{a_{4(n-\nu-1)+1}(b_{4\nu+2} - b_{4\nu}) - a_{4(n-\nu-1)}(b_{4\nu+3} - b_{4\nu+1})\}$$

$$= \sum_{\nu=0}^{\nu=n'-1} (b_{4\nu} - b_{4\nu+2})(a_{4(n-\nu-1)} - a_{4(n-\nu-1)+1})$$

$$+ \sum_{\nu=n'}^{\nu=n-1} (b_{4\nu} - b_{4\nu+2})(a_{4(n-\nu-1)} - a_{4(n-\nu-1)+1}),$$

or numerically

$$< \sum_{\nu=0}^{\nu=n'-1} 4\varepsilon_{4\nu+2} \cdot \varepsilon_{4(n-\nu-1)} + \sum_{\nu=n'}^{\nu=n-1} 4\varepsilon_{4\nu+2} \cdot \varepsilon_{4(n-\nu-1)}$$

$$< c\varepsilon_{4(n-n')} + c'\varepsilon_{4n'}, \qquad \text{(I)}$$

where

$$c \geqq \sum_{\nu=0}^{\nu=n'-1} 4\varepsilon_{4\nu+2}, \quad c_1' \geqq \sum_{\nu=n'}^{\nu=n-1} 4\varepsilon_{4(n-\nu-1)},$$

$n' = \frac{1}{2}(n-2)$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

Each of the terms in (I) is of the same order of magnitude as $\dfrac{1}{4n(\lg 4n)^\lambda}$.

The same reasoning which we used with the $(4n)$th term can be applied to the $(4n+1)$th, $(4n+2)$th, $(4n+3)$th term. Thus in the product of the two conditionally convergent series S_1 and S_2 each term is numerically less than the corresponding term of a series known to be absolutely convergent. Hence the product of S_1 and S_2 is absolutely convergent.

As an example, we give the two series T_1 and T_2, obtained respectively by dropping the parentheses from the following series:

$$T_1' \equiv \sum_{v=0}^{v=\infty} \left(\frac{1}{4v'+1} - \frac{1}{4v'+4} + \frac{1}{4v'+1} - \frac{1}{4v'+4} \right),$$

$$T_2' \equiv \sum_{v=0}^{v=\infty} \left(\frac{1}{4v'+4} + \frac{1}{4v'+4} - \frac{1}{4v'+1} - \frac{1}{4v'+1} \right),$$

where $\frac{1}{2} < r \leqq 1$ and $\frac{1}{2} < s \leqq 1$. T_1 and T_2 are each conditionally convergent; their product is absolutely convergent.

§ 2.

The behavior of infinite series with respect to the fundamental laws of algebra may be considered under two heads: An inquiry into the validity of the laws (1) when applied to the *terms* of an infinite series, (2) when applied to the infinite series *themselves*.

The first inquiry has led to the result that the associative law can always be applied to the terms of a convergent infinite series, but that the commutative law can be applied, in general, only to the terms of an absolutely convergent series.

The second inquiry has been made for the addition (and subtraction) of infinite series but, so far as we have seen, not for their multiplication with each other.

The product of

$$U \equiv \sum_{n=0}^{n=\infty} u_n \quad \text{and} \quad V \equiv \sum_{n=0}^{n=\infty} v_n$$

has been defined by Cauchy to be

$$\sum_{n=0}^{n=\infty} (u_0 v_n + u_1 v_{n-1} + \cdots + u_n v_0).$$

Law of Association.—This law can be applied without limitation to the multiplication of series.

To show this, let $W \equiv \sum_{n=0}^{n=\infty} w_n$, where w_n, as well as u_n and v_n given above, are finite constant numbers, real or complex. Then we have $(UV)W = U(VW)$, for the $(n+1)$th term in the product $(UV)W$ is

$$
\begin{aligned}
&(u_0v_n + u_1v_{n-1} + \cdots + u_nv_0)w_0 \\
&+ (u_0v_{n-1} + u_1v_{n-2} + \cdots + u_{n-1}v_0)w_1 \\
&+ (u_0v_{n-2} + \cdots + u_{n-2}v_0)w_2 \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+ (u_0v_0)w_n.
\end{aligned}
$$

The $(n+1)$th term of the product $U(VW)$ is

$$
\begin{aligned}
&u_0(w_0v_n + w_1v_{n-1} + \cdots + w_nv_0) \\
&+ u_1(w_0v_{n-1} + w_1v_{n-2} + \cdots + w_{n-1}v_0) \\
&+ u_2(w_0v_{n-2} + \cdots + w_{n-2}v_0) \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+ u_n(w_0v_0).
\end{aligned}
$$

These two expressions for the $(n+1)$th term, for any positive integral value of n, no matter how large, are seen to be identical as soon as we give our assent to the following two statements:

1. For $n > q$, where q is any positive finite number, we have always

$$u_p(w_0v_{n-p} + w_1v_{n-p-1} + \cdots + w_{n-p}v_0) = u_pw_0v_{n-p} + \cdots + u_pw_{n-p}v_0.$$

2. For $n > q$, we are allowed to commute the terms obtained by removing the parentheses, provided of course that no terms be dropped from the total aggregate and no new terms be admitted to it. It will be seen that this special case does not contradict our previous statement that the commutative law is not, in general, applicable to the terms of series not absolutely convergent.

The first expression for the $(n+1)$th term assumes the form of the second if in the first we perform in each row the indicated multiplication, then add the columns from left to right, and factor.

Since the $(n+1)$th term in $(UV)W$ is the same as the $(n+1)$th term in $U(VW)$, no matter what positive integral value be assigned to n, it follows that the two products are identical. Thus the associative law is always obeyed.

Law of Commutation.—Cauchy's definition makes the product $\Sigma u_n \cdot \Sigma v_n$ the same as the product $\Sigma v_n \cdot \Sigma u_n$, so that the commutative law holds for two factor series. Being permitted to assume the associative law, it follows easily that

the commutative law is valid for three or more factor series. Thus,

$$UVW = U(VW) = U(WV) = UWV = (UV)W = (VU)W = VUW = V(WU) = VWU = W(UV) = WUV = W(VU) = WVU.$$

Law of Distribution.—That $U(V+W) = UV + UW$ can be shown in a manner similar to our proof of the associative law, *viz:* find the $(n+1)$th term of $U(V+W)$ and the $(n+1)$th term for $UV + UW$. Then assuming the commutative and distributive laws to hold for the aggregate of terms involved in one of the expressions, change it into the other.

§ 3.

Proceeding as does Pringsheim, let

$$f(x) = \sum_{\nu=0}^{\nu=\infty} (-1)^\nu a_\nu x^\nu, \qquad \text{(II.)}$$

where a_ν is of the same order of magnitude as ν^{-r}, $\frac{1}{2} < r < 1$. Here $x = -1$ is a singular point on the circle of convergence and $f(-1)$ is an infinity of the same order as $(1-1)^{-(1-r)}$. If the series (II) is raised to the positive integral power p, then the sum of the resulting series, for $x = -1$, is of the same order of infinity as $(1-1)^{-p(1-r)}$. If the power $p \geqq \frac{1}{1-r}$, then the order of infinity is not lower than the first. But, for $x = +1$, series (II) becomes a special case of S_1 of § 1. Hence the pth power of S_1 is divergent, when a_ν is of the same order of magnitude as ν^{-r}.*

In the same way it can be shown that the pth power of series S_2 is divergent, when b_ν is a magnitude of the same order as ν^{-r}, $\frac{1}{2} < r < 1$, and $p \geqq \frac{1}{1-r}$. But $S_1 \cdot S_2$ was shown to be absolutely convergent. We have $S_1 \cdot S_2 \cdot S_1 \cdot S_2 \cdots$ (to p pairs of factors) $= (S_1S_2)(S_1S_2) \cdots$ (to p parentheses). Hence the product of these $2p$ series is absolutely convergent. But, by the associative and commutative laws, this product is equal to $S_1^p \cdot S_2^p$. Thus, S_1^p and S_2^p are two divergent series whose product is absolutely convergent. Observe that, no matter how much p is in excess of $\frac{1}{1-r}$—that is, no matter how high a power of S_1 and S_2 is taken—we have, for a given value of r, always an absolutely convergent product result-

* Pringsheim, *loc. cit.*, pp. 409–411.

ing from the multiplication of S_1^r by S_2^r. Special example: $T_1^{\frac{1}{2}} \cdot T_2^{\frac{1}{2}}$ is absolutely convergent, but $T_1^{\frac{1}{2}}$ and $T_2^{\frac{1}{2}}$ are each divergent when $r \leq \frac{2}{3}$ and $s \leq \frac{2}{3}$.

In the divergent series S_1^r the terms increase without limit in numerical value, as v increases without limit. The same is true of S_2^r. Herein lies the difference between this pair of divergent series yielding an absolutely convergent product, and the pair given by Pringsheim.* In the latter the terms of the divergent series remain finite as v increases indefinitely.

From the relation $S_1 S_2 S_1 S_2 \cdots = S_1^r \cdot S_2^r$ we see that there are cases in the multiplication of series in which divergent series may be used with safety—the sum of the final product series being convergent and equal to the product of the sums of the initially given convergent factor series, even when the product of *some* of the given factor series is divergent.

If two or more convergent series, when multiplied together, yield a convergent product series, then the sum of this product series is equal to the product of the sums of the factor series.

This theorem was proved by Abel for the case of two factor series,† and his method of proof is applicable to the general case. The extension is obvious.

COLORADO COLLEGE, COLORADO SPRINGS,
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CONCERNING THE CLASS OF A GROUP OF ORDER p^m THAT CONTAINS AN OPERATOR OF ORDER p^{m-2} OR p^{m-3}, p BEING A PRIME.

BY DR. W. B FITE.

(Read before the American Mathematical Society, December 28, 1901.)

IF a non-abelian group of order p^m contains an operator of order p^{m-1} it is of the second class.‡ It is the object of

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* *Loc. cit.*, p. 409.

† *Oeuvres complètes* de N. H. Abel, Tome Premier, 1839, "Recherche sur la série $1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \ldots$," Theorem VI.

‡ Burnside, Theory of Groups, p. 76. If we form the group of cogredient isomorphisms G' of G, then the group of cogredient isomorphisms G'' of G', and so on we finally come either to identity or to a group that has no invariant operators except identity, and is therefore simply isomorphic with its group of cogredient isomorphisms. The groups for which this process leads to identity are classified according to the number of these successive groups of cogredient isomorphisms.

the present note to show that a group of order p^m which contains an operator of order p^{m-2} is of class k, where $k \geqq 3$, if p is an odd prime; and that $k \geqq 4$ if the group is of order p^m and contains an operator of order p^{m-3}, where p is a prime greater than 3.

Let G be a group of order p^m that contains an operator of order p^{m-a}, where $a \geqq \frac{1}{2}m$ and p is an odd prime. Assume that in all groups of order p^{m_1} that contain an operator of order $p^{m_1-a_1}$, where $a_1 \geqq \frac{1}{2}m_1$ and $a_1 < a$, the p^{a_1} power of every operator is invariant. Let A be an operator of G of order p^{m-a}. We can assume that G contains no operator of order greater than p^{m-a}.

G contains an invariant subgroup G_1 of order p^{m-1} that contains A.* If $G \equiv \{G_1, B\}$, then $A^{p^{a-1}}B^p = B^pA^{p^{a-1}}$, since in accordance with our assumption $A^{p^{a-1}}$ is invariant in G_1. Now

$$B^{-1}AB = AC,$$

where C is an operator of G_1.

$$B^{-1}A^{p^{a-1}}B = (AC)^{p^{a-1}} = A^{p^{a-1}(1+b)},$$

since the p^{a-1} power of every operator of G_1 is in $\{A\}$

$$B^{-p}A^{p^{a-1}}B^p = A^{p^{a-1}(1+b)^p} = A^{p^{a-1}}.$$

Therefore

$$(1+b)^p \equiv 1 \pmod{p^{m-2a+1}} \text{ and } b \equiv 0 \pmod{p^{m-2a}}.$$

If

$$m - 2a = 0, \qquad b \equiv 0 \pmod{p^{m-2a+1}}.$$

Therefore $B^{-1}A^{p^a}B = A^{p^a}$, and the order of G' is equal to, or less than, p^{2a}.

This result was reached on the assumption that a similar result holds for groups of order p^{m_1} that contain an operator of order $p^{m_1-a_1}$, where $a_1 < a$. Now this result holds when $a_1 = 1$. Therefore it holds for all values of $a \geqq \frac{1}{2}m$.

If $a = 2$, the order of G' is equal to, or less than, p^4. It remains to be shown that G' cannot be of the third class. If it were of the third class it would be of order p^4 and contain an operator of order p^2, but none of a higher order.

* Frobenius, *Berliner Sitzungsberichte*, 1895, p. 173. Burnside, *Proc. London Math. Soc.*, vol. 26 (1895), p. 209.

In this case if B^p were not in $\{A^1\}$, G' would be generated by two independent operators of order p^2. But there is no such group.*

If $B^p = A^{lp}$ we must have $l \equiv 0 \pmod{p}$, since otherwise A^p would be invariant in G, and the order of G' would be less than p^4. But if we had $l \equiv 0 \pmod{p}$, G' would be generated by an operator of order p^2 and two of order p. This is impossible.† Therefore:

A group of order p^m, where p is an odd prime, that contains an operator of order p^{m-2} is of class k, where $k \geqq 3$.

Suppose now that $a = 3$ and $p > 3$. The order of G' is equal to or less than p^6. If this order equals p^6, G' contains an operator of order p^3, but none of higher order. If G_2 is a subgroup of G_1 of order p^{m-2} that contains A, there must be an operator D in G_2 of order p that is not contained in $\{A\}$, since otherwise G_2 would contain only one subgroup of order p and therefore would be cyclic.‡ But we have supposed that G contains no operator of order higher than p^{m-3}. Therefore if C were an operator of G_1 that is not in $\{G_2\}$ and that corresponds to an operator of G' of order p^3,§ we should have $C^p = A^{lp}$, where $l \not\equiv 0 \pmod{p}$, or $C^p = A^{lp}D$. In the former case, A^{p^2} would be invariant in G and the order of G' would be less than p^6. In the latter, $k \geqq 4$. If $B^{p^2} = A^{lp^2}$, where $l \not\equiv 0 \pmod{p}$, A^{p^2} would be invariant in G, and the order of G' would be less than p^6. Therefore if the order of G' is p^6, we need to consider only the cases in which G' is generated either by two independent operators of order p^3 or by an operator of order p^3 and one or more operators of orders less than p^3. The latter is impossible, and the former is possible only in case G' is of class k, where $k \geqq 3$, since if $k > 3$ one of the successive groups of cogredient isomorphisms would have one generator of order greater than the order of any other generator. This is impossible.

If the order of G' is less than p^5, then $k \geqq 4$. We consider therefore the case in which G' is of order p^5. If A^{p^2} is not invariant in G, G' has an operator of order p^3 and therefore $k \leqq 4$. Accordingly, we suppose that A^{p^2} is invariant in G. Then G' has an operator of order p^2. Now

* Young, *Amer. Jour. of Math.*, vol. 15 (1893), pp. 124–178; Hölder, *Math. Annalen*, vol. 43 (1893), pp. 371–412.

† This is proved in an article offered for publication to the *Transactions of the American Mathematical Society*.

‡ Burnside, Theory of Groups, 1897, p. 73.

§ We assume that G and G' are arranged in an $(a, 1)$ isomorphism, the a invariant operators of G corresponding to identity of G'.

every group of the fourth class, of order p^5, where $p > 3$, that has an operator of order p^2 is generated by an operator of order p^3 and three of order p.* No such group can be a group of cogredient isomorphisms. Therefore:

A group of order p^m, where p is a prime ≥ 3, that contains an operator of order p^{m-3} is of class k, where $k \geqq 4$.

CORNELL UNIVERSITY.

PROOF THAT THE GROUP OF AN IRREDUCIBLE LINEAR DIFFERENTIAL EQUATION IS TRANSITIVE.

BY DR. SAUL EPSTEEN.

(Read before the American Mathematical Society, December 28, 1901.)

WE will define with Frobenius† the linear differential equation

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0 \tag{1}$$

to be irreducible when it has no integral in common with a differential equation of the same character, but of a lower order.

Picard has shown‡ that for the equation (1) a linear homogeneous group

$$Y_i = \sum_k a_{ik} y_k \tag{2}$$

plays the same rôle as the group of substitutions plays in the Galois theory of algebraic equations.

It is the object of this brief paper to show that the group (2) of the equation (1) will be transitive§ when the equation is irreducible and intransitive when the equation is reducible, and we shall carry the proof out on lines analogous to the corresponding theorem in algebra.‖

The basis of our proof is the Lagrange-Vessiot theorem,¶ that if the rational differential function $S(y)$ of the in-

* Bagnera, *Ann. di Matematica*, ser. 3, vol. 1 (1898), p. 218.
† Frobenius, *Crelle*, vol. 76.
‡ Picard, *Comptes rendus*, 1883.
§ Lie-Engel, Transformationsgruppen, I., ch. 13.
‖ Netto, Substitutionstheorie, § 154.
¶ E. Vessiot, *Annales de l'Ec. Norm. Sup.*, 1892.

tegral is invariant under the group of the differential function $R(y)$, then S can be expressed rationally by means of R, the coefficients of the equation (1), their derivatives, and x, that is: $S =$ Rational Function $(R, R', \cdots, p_1, \cdots, p_n, p_1', \cdots, p_n', x)$.

Suppose that the group of the equation (1) is intransitive, or in other words that we can transform the system of integrals

$$Y_1, \cdots, Y_m \quad \text{into} \quad y_1, \cdots, y_m \qquad (m < n)$$

by means of the group, but not

$$Y_{m+1}, \cdots, Y_n \quad \text{into} \quad y_{m+1}, \cdots, y_n.$$

Let us now form the linear differential equation of the mth order which has $Y_1, \cdots, Y_m$ as a fundamental system of integrals

$$\frac{d^m y}{dx^m} + q_1 \frac{d^{m-1} y}{dx^{m-1}} + \cdots + q_m y = 0.$$

The coefficients of this equation in terms of the integrals are

$$q_\lambda = - \frac{\begin{vmatrix} Y_1 & Y_2 & \cdots & Y_m \\ Y_1' & Y_2' & \cdots & Y_m' \\ \vdots & \vdots & & \vdots \\ Y_1^{(m-\lambda-1)} & Y_2^{(m-\lambda-1)} & \cdots & Y_m^{(m-\lambda-1)} \\ Y_1^{(m-\lambda+1)} & Y_2^{(m-\lambda+1)} & \cdots & Y_m^{(m-\lambda+1)} \\ \vdots & \vdots & & \vdots \\ Y_1^{(m-1)} & Y_2^{(m-1)} & \cdots & Y_m^{(m-1)} \end{vmatrix}}{\begin{vmatrix} Y_1 & \cdots & \cdots & Y_m \\ Y_1' & \cdots & \cdots & Y_m' \\ \vdots & & & \vdots \\ Y_1^{(m-1)} & \cdots & \cdots & Y_m^{(m-1)} \end{vmatrix}} \qquad (\lambda = 1, \cdots, n)$$

Our hypothesis is that the group (2) will, by a suitable choice of the parameters, bring the system of solutions $Y_1, \cdots, Y_m$ exactly into $y_1, \cdots, y_m$; that is to say, the coefficients q of the equation (3) are invariant under the group of (1) and consequently, by the Lagrange-Vessiot theorem,

$$q_\nu = \text{Rat. Funct.}\ (p_1, \cdots, p_n, p_1', \cdots, p_n', x).$$

The equation (1) has therefore the m integrals $Y_1, \cdots, Y_m$ $(m < n)$ in common with the equation (3). But the equation (3) is linear, of lower order than (1), and, as we have

just seen, its coefficients are rational. It follows from the definition that the equation (1) is reducible.

Conversely, it is now easy to see that if the equation (1) is reducible, its group must be intransitive.

We are therefore enabled to enunciate the following theorem, exactly analogous to a theorem in algebra:

The group of an irreducible linear differential equation is transitive.

GÖTTINGEN,
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LINES OF LENGTH ZERO ON SURFACES.

BY DR. L. P. EISENHART.

(Read before the American Mathematical Society, December 28, 1901.)

By definition, the directions of the double system of lines of length zero on a surface are given by equating to zero the expression for the square of the linear element of the given surface, that is, in the Gauss notation,

$$Edu^2 + 2Fdudv + Gdv^2 = 0. \tag{1}$$

Since these lines are always imaginary and since the lines of curvature of a real surface are always real there is no real surface whose lines of curvature are of length zero. It is well known that lines of length zero form a conjugate system upon a minimal surface and further that this is a characteristic property of such surfaces. In order that asymptotic lines be of length zero it is necessary and sufficient that the two fundamental forms of the surface be proportional. Hence the sphere is the only real surface whose asymptotic lines are of length zero; they are the imaginary rectilinear generatrices of the sphere. When the parameters of these lines are taken as conjugate imaginaries and the sphere is of radius unity and with center at the origin, the cartesian coördinates have the following expressions:

$$x = \frac{u+v}{1+uv}, \quad y = i\frac{v-u}{1+uv}, \quad z = \frac{uv-1}{1+uv}. \tag{2}$$*

We propose now the following problem:

* Darboux, Leçons, I, p. 245.

For what surfaces is the spherical representation of lines of length zero on the surface given by lines of length zero on the sphere?

When $u = \text{const.}$, $v = \text{const.}$ on the sphere are lines of length zero, the formulæ (2) give the expressions for the direction cosines of the normal to the surface S, hence the surface is the envelope of the plane whose equation is

$$(3) \qquad (u+v)x + i(v-u)y + (uv-1)z + \xi = 0$$

and the cartesian coördinates of points on S are given by*

$$(4) \qquad x + iy = -vz - p, \quad x - iy = -uz - q,$$

$$z = \frac{\xi - pu - qv}{1 + uv},$$

where

$$p, q = \frac{\partial \xi}{\partial u}, \frac{\partial \xi}{\partial v}.$$

The square of the linear element of S is

$$ds^2 = (zdu + dq)(zdv + dp),$$

or, on writing

$$r, s, t = \frac{\partial^2 \xi}{\partial u^2}, \frac{\partial^2 \xi}{\partial u \partial v}, \frac{\partial^2 \xi}{\partial v^2}$$

$$ds^2 = [(z+s)du + tdv][rdu + (z+s)dv].$$

From this it follows that the necessary and sufficient condition that the lines $u = \text{const.}$, $v = \text{const.}$ on S be of length zero is

$$(z+s)r = 0, \quad (z+s)t = 0;$$

hence either

$$(5) \qquad z + s = 0, \qquad \text{or} \qquad r = t = 0.$$

The former is a characteristic equation of minimal surfaces.† To discuss the latter we make use of the formulæ‡

$$2\rho_1 = \xi - pu - qv + (1+uv)(s + \sqrt{rt}),$$

$$2\rho_2 = \xi - pu - qv + (1+uv)(s - \sqrt{rt}),$$

* Darboux. Leçons, I, p. 246.
† Ib., p. 297.
‡ Ib., p. 246.

where ρ_1 and ρ_2 denote the principal radii of S. For the second of the conditions (5) this gives $\rho_1 = \rho_2$, that is, S is a sphere, when real. Combining these results we have that *minimal surfaces and spheres are the only real surfaces for which the spherical representation of lines of length zero is the system of rectilinear generatrices of the sphere.*

In order to determine what surfaces have the lines of length zero on the sphere for spherical representation of its asymptotic lines we note that for the parameters used previously the equation of the asymptotic directions is*

$$rdu^2 + 2(s + z)dudv + tdv^2 = 0.$$

Hence, in order that the u and v lines may be asymptotic for the surface, the condition is

$$r = t = 0,$$

which as we have seen characterizes the sphere. Again, in order that the parametric lines may form a conjugate system we get from the above equation the condition

$$s + z = 0,$$

that is minimal surfaces. Recalling the above theorem we have the following:

In order that the asymptotic lines or a conjugate system on a surface may be represented upon the sphere by its imaginary generatrices, they must be lines of length zero on the surface.

PRINCETON,
November, 1901.

SOME PROPERTIES OF POTENTIAL SURFACES.

BY DR. EDWARD KASNER.

(Read before the American Mathematical Society, April 27, 1901.)

IN a previous paper, published in the BULLETIN, volume 7, pages 392–399, the author studied the algebraic curves $\varphi(x\,y) = 0$ defined by the condition $\varphi_{xx} + \varphi_{yy} \equiv 0$. Two classes of characteristic properties were obtained, the first translating directly the differential equation and the second arising from the well-known connection between harmonic

* Darboux, Leçons, I, p. 246.

functions and functions of the complex variable. The present paper extends some of the results to the corresponding type of surfaces,* *i. e.*, to the surfaces expressed in rectangular coördinates by an equation $\varphi(x, y, z) = 0$ where φ is a rational integral solution of the potential equation

$$\Delta\varphi \equiv \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0. \tag{1}$$

The results obtained in § 3, however, apply not merely to algebraic but to all analytic potential surfaces.

§ 1. *Apolar Properties.*

The algebraic potential surfaces are characterized most simply by their relation to the imaginary circle at infinity.† Considering the latter as a degenerate class quadric, its equation in rectangular plane coördinates may be written

$$Q \equiv u^2 + v^2 + w^2 = 0. \tag{2}$$

In homogeneous coördinates the condition that a quadric

$$u_P^2 \equiv \sum P_{ik} u_i u_k = 0 \quad (i, k = 1, 2, 3, 4)$$

is apolar to a surface

$$\Phi \equiv a_x^n = 0$$

is expressed by the identical vanishing of the covariant

$$S \equiv a_P^2 a_x^{n-2} \equiv \sum P_{ik} \frac{\partial^2\Phi}{\partial x_i \partial x_k}.$$

Applying this to rectangular coördinates and assuming the quadric to be Q, this covariant takes the form

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}.$$

Comparing this with the defining equation (1) we have

* The focal properties of the curves do not seem capable of direct generalization. The n-dimensional extensions of the results obtained in this paper are in general obvious.

† That there must be such a relation is seen a priori from the fact that the defining differential equation is invariant under the conform group, which contains the group of euclidean geometry as a subgroup.

I. *All potential surfaces are apolar to the imaginary circle at infinity considered as a degenerate quadric; conversely, any surface apolar to this circle is a potential surface.*

The imaginary element which enters into this characterization may be eliminated by considering the polar quadrics of the surface, which, from the definition of the apolar relation, must themselves be apolar to Q. If a quadric surface is apolar to Q it must intersect the plane at infinity in a conic which, as a curve of second order, is apolar to the imaginary circle considered as a curve of second class, and therefore contains triples of points mutually conjugate with respect to the circle. It follows that the asymptotic cone of the quadric contains among its generators triples of mutually orthogonal lines. The potential quadrics are thus identical with what may be termed the rectangular or equilateral hyperboloids* from their analogy to the rectangular hyperbolas. Theorem I may now be restated

II. *The polar quadric of any point with respect to a potential surface is a rectangular hyperboloid; conversely, if all the polar quadrics of a surface are of this species, it is a potential surface.*

Since the polar quadrics of a polar surface coincide with the polar quadrics of the original surface. we have

III. *The polar surfaces of a potential surface are also potential surfaces.*

§2. *Special Potential Surfaces.*

A relation between the potential curves and surfaces presents itself when the surface considered is cylindrical. The equation of such a surface may be assumed in the form $\varphi(x, y) = 0$, so that $\Delta\varphi$ reduces to $\varphi_{xx} + \varphi_{yy}$. Since the vanishing of this expression is the condition for a potential curve, the result may be stated

IV. *A cylindrical surface belongs to the class of potential surfaces when, and only when, its right section is a potential curve.*

This relation may be generalized by projecting the imaginary circle Q into an arbitrary conic and referring to theorem IV of the article cited above on potential curves.

IV′. *If a conical surface with its vertex V in the plane of a conic C is apolar to the latter (considered as a degenerate class quadric), then any plane passing through the polar line of V with respect to C intersects the conical surface in a curve apolar to the point pair cut out upon C; conversely, etc.*

* An equation of the second degree in x, y, z represents such a quadric when the sum of the coefficients of x^2, y^2, and z^2 equals zero, as may be verified directly from the condition $\Delta\phi = 0$. For a complete discussion of these quadrics cf. H. Vogt, "Ueber ein besonderes Hyperboloid," *Crelle*, vol. 86 (1879), pp. 297-316.

If a potential cylindrical surface degenerates into a set of planes, the right section will consist of a set of lines forming a potential curve. Applying the theorem* concerning the asymptotes of potential curves, it follows that the lines must be concurrent and disposed symmetrical about their common point.

V. *A set of planes having a common perpendicular plane constitute a potential surface when, and only when, the planes intersect in a common line and are disposed symmetrically about it.*

Another type† of degenerate potential surfaces formed of a number of planes is obtained by adding to the above set the common perpendicular plane. For if $\varphi(x, y)=0$ represents the original set of planes, the new set may be represented in the form $z\varphi(x, y) = 0$, and the Laplacian of the first member is $z(\varphi_{xx} + \varphi_{yy})$.

Among the potential functions, those which are homogeneous play an important rôle. The corresponding surfaces are cones whose characteristic property may be deduced from the following

LEMMA. *If a cone S_n of nth order is apolar to a plane curve C_k of kth class, considered as a (degenerate) surface C_k' of kth class, then S_n intersects the plane of C_k in a curve S_n' which is apolar to C_k; the converse also holds.*

In the proof any system of homogeneous coördinates may be assumed since the relations considered are projective. Take then the vertex of the cone as one vertex of the fundamental tetrahedron and let the other three be taken in the plane of C_k. The equation of S_n is then of the form $S(x_1, x_2, x_3) = 0$, and that of C_k' is $C(u_1, u_2, u_3) = 0$. The equations of the related curves S_n' and C_k are the same as those of the surfaces, since x_1, x_2, x_3 and u_1, u_2, u_3 may be interpreted, independently of x_4 and u_4, as trilinear coördinates in the plane $u_4 = 0$ which is the plane of C_k. The condition of apolarity between the surfaces S_n, C_k' is thus identical with the condition of apolarity between the curves S_n', C_k. This is the result expressed in the lemma.

Applying this to the case when C_k becomes Q, the imaginary circle at infinity, we have‡

XI. *A conical surface is a potential surface when, and only*

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* Briot and Bouquet, Théorie des fonctions elliptiques, p. 227.

† Cf. the types referred to in Thomson and Tait's Natural Philosophy, 2d ed., § 781.

‡ This is equivalent to Clifford's theorem concerning nodal curves, "On the canonical form of spherical harmonics," Works, p. 234. Cf. W. Thomson, "On the general canonical form of a spherical harmonic of nth order," *Report 41st Meeting Brit. Assoc. Adv. Science*, 1871, p. 10.

when, it intersects the plane at infinity in a curve apolar to the imaginary circle.

§ 3. *Nodes and Multiple Curves.*

The preceding results relating to algebraic potential surfaces may be applied to the more general class of analytic potential surfaces. Referred to a system of rectangular coordinates with the origin situated on the surface, the equation an analytic surface is of the form

$$\varphi \equiv v_1 + v_2 + \cdots + v_k + \cdots = 0,$$

where v_k is a homogeneous function of degree k in x, y, z. Since

$$\Delta\varphi = \Delta v_1 + \Delta v_2 + \cdots + \Delta v_k + \cdots$$

and since no two terms in the second member are of the same degree, it follows that for a potential surface Δv_k must vanish identically for every value of k.

If now the origin is a node of order $k-1$,

$$v_1 \equiv 0,\ v_2 \equiv 0,\ \cdots,\ v_{k-1} \equiv 0,$$

and $v_k = 0$ is the equation of the tangent cone. But from the above $\Delta v_k \equiv 0$, *i. e.*, the cone must be a potential surface.

VII. *If a potential surface has a node of order $k-1$, the tangent cone is a potential cone of kth order. When the node is simple the cone is of the second order and contains triples of mutually orthogonal generators.*

If the analytic surface considered has a multiple curve of order $k-1$, and the origin is taken anywhere on the curve, $v_k = 0$ represents the k planes tangent to the different sheets of the surface. Since these planes all pass through the line tangent to the multiple curve, theorem IV may be applied; therefore *

VIII. *If a potential surface has a multiple curve of order $k-1$, then the k sheets passing through the curve intersect each other at angles equal to $2\pi/k$; when $k=2$, the two sheets are orthogonal.*

The decomposable surface $\varphi\psi = 0$, formed by two potential surfaces $\varphi = 0$, $\psi = 0$ is not in general a potential surface. The Laplacian of a product may be developed

* For the corresponding theorem concerning the multiple points of potential curves, see Briot and Bouquet, *l. c.*, p. 225; Thomson and Tait, Natural philosophy, 2d ed., § 780, where reference is made to an earlier paper by Rankine, "Summary of the properties of certain stream lines," *Phil. Mag.*, 1864.

$$\Delta(\varphi\psi) = \varphi\Delta\psi + \psi\Delta\varphi + 2\Omega,$$

where

$$\Omega = \frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial z}\frac{\partial\psi}{\partial z}.$$

As is well known, Ω vanishes only when the two families $\varphi = \text{const}$ $\psi = \text{const}$, are orthogonal. Therefore

IX. *If a pair of potential surfaces $\varphi = 0$, $\psi = 0$ combined form a potential surface, the families $\varphi = \text{const.}$, $\psi = \text{const.}$ are orthogonal.*

The projective generalization of this result is

IX′. *If a surface apolar to a conic decomposes into two surfaces apolar to the same conic, then the tangent planes to the latter surfaces at any point in their intersection cut the plane of the conic in a pair of lines conjugate with respect to the conic.*

A MODERN ENGLISH CALCULUS.

An Elementary Treatise on the Calculus, with Illustrations from Geometry, Mechanics and Physics. By GEORGE A. GIBSON, M.A., F.R.S.E., Professor of Mathematics in the Glasgow and West of Scotland Technical College. London, Macmillan & Co., 1901. 12mo, pp. xix + 459.

IN the year 1891 Harnack's Elements of the Differential and Integral Calculus, which appeared in Leipzig in 1881, was translated into English. This book gave the first systematic presentation in the English language of the leading principles of modern analysis in their relation to the foundations of the infinitesimal calculus. While not wholly free from errors, and sometimes difficult to read, owing to inadequate exposition of details, the book is nevertheless conceived in the spirit of modern mathematics and it lays stress on those principles of analysis which are essential for a rigorous development of the calculus.

The first book of English origin to treat the calculus from a modern standpoint was Lamb's Infinitesimal Calculus,* published in 1897. This is an excellent treatise and any later work on the calculus, of modern tendencies, must have many points of contact with it.

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*A notice of this book by the present writer appeared in *Science*, new series, vol. 7 (1898), No. 176, p. 678.

The year that has just closed has witnessed the appearance of a new text-book on this subject. by Professor Gibson, who, like Professor Lamb, is a teacher in a technical school and was led to write the book by the desire to afford to technical students the training in mathematics which will be useful to them in their profession. The author's conception of what is a useful training for such students differs widely from the ideas that prevail among many teachers in technical schools, who think they have done their whole duty by their class when they have taught them to perform the formal work of differentiation and integration, and to compute centers of gravity and moments of inertia from the formulas. From the start Professor Gibson lays stress on the conceptions and the methods of the calculus. The notion of the function is set forth, not in a line or two, but in over a page of text, well illustrated by examples, and ending with a formal definition which is both clear and accurate. The meaning of the symbol $f(x)$ is fully explained. In Chapters II and III the subject of graphs is treated at length. These chapters will be welcome to teachers of the calculus as a convenient place to which to refer students whose previous training in analytic geometry has been irregular and who wish to acquire in a brief space of time that part of the subject which is essential for beginning the study of the calculus.

Chapter IV is entitled Rates; Limits. The idea of the rate of change of one variable with respect to another is set forth in such a manner that the beginner cannot help feeling the necessity of the last step—the introduction of the limit as the simplest and most natural way of *defining* the actual rate or velocity, when the change of the dependent variable is not a uniform one. A general explanation of a limit follows and then two formal definitions of a limit are given. The first is accurate and easily intelligible to the beginner. The second is the ϵ definition. But the author is judicious in his use of ϵ proofs, none occurring in the early chapters and only a few at places in the later chapters where no other proof is so simple. The treatment of the conception of a function's becoming infinite is too brief, and the expression "converges to ∞" is to be deprecated, no matter how carefully the meaning which the author attaches to it may be defined. If the student is taught to read the notation: $x = \infty$ as "when x becomes infinite," the very language that he uses conduces to the formation of right ideas. This is the more important since such ideas are not formed from definitions and precept, but

from practice and example. For the same reason it is desirable to read $\mathrm{L}_{x=a} f(x)$, not as "limit of $f(x)$ for x equal to a," but as "limit of $f(x)$ when x approaches a." The chapter, taken as a whole, is a strong one and one that every beginner in the calculus will do well to read. The examples are a particularly noteworthy feature

Chapter V deals with continuity and certain special limits, in particular

$$\lim_{m=\infty} \left(1+\frac{1}{m}\right)^m.$$

In these five chapters, occupying 100 pages, not only the fundamental conceptions of the calculus are treated, but all of the special limits are evaluated which are needed in the differentiation of the elementary functions. Chapters VI and VII, 48 pages, are devoted to differentiation, Chapter VIII, 12 pages, to physical applications. This arrangement has the advantage of being systematic and makes it easy for the student at any later time to turn to the particular topic to which he wishes to refer. It has the disadvantage, however, of not presenting the subject from its most attractive and from its most accessible side. The reviewer believes that it is preferable to begin with the problem, already treated to some extent in analytic geometry, of finding the tangent to a curve and then, after explaining the notion of the function and the method of representing it by a graph, to take up a few simple differentiations and to illustrate the use of the derivative by numerous problems in velocities, maxima and minima, and graphs. Consider to how great an extent elementary mathematics is formal. School algebra consists largely, at least as at present taught, in learning rules and applying them to examples that are precise copies of illustrative ones worked out in the text-book, and it thus sinks almost to the level of Latin composition. The "originals" of geometry, and many parts of analytic geometry, especially loci problems, afford real training in mathematical reasoning. But the formal tendency is still predominant when the student begins calculus. Problems in velocities and maxima and minima present at the outset a means of making the student think about what he is doing, provided that they are so chosen that he must formulate his data. Thus the problem, "A point moves in such a manner that the space traversed is given by the formula $s = 120 - 16t^2$; find its velocity at the end of two seconds," can have no higher value than to encourage a dull student

to begin work on the exercises assigned. On the other hand the problem, "A vessel is anchored in three fathoms of water and the cable passes over a sheave in the bowsprit, which is six feet above the water. If the cable is hauled in at the rate of a foot a second, how fast is the vessel moving through the water when there are five fathoms of cable out?" —given out after the class has had some practice on easier problems, but not preceded by a similar problem worked for the student, is of value in making the student think for himself. The same is true of problems in maxima and minima, those that require the student to put into equations data that are given him without formulas affording effective training in the principles to which the calculus owes its origin. In their essential features these problems are of the nature of physical applications. They train the student to use his calculus from the start without involving the technical difficulties of many of the problems which Professor Perry considers in his book, "Calculus for Engineers." We are in hearty sympathy with Professor Perry as regards the objects which he has in view, for we believe that calculus should be brought home to the student and that he should through many and varied problems be brought to feel that calculus stands in vital relation to the phenomena of every day life. No more fatal criticism on a course in calculus can be made than that which is contained in the remark of the student who says that he does not see what calculus is for. Professor Perry's book contains much that can be so far freed from technical difficulties as to become available for the general student, and the same is true of the books of Autenheimer * and Nernst and Schoenflies.† As a means to the same end, too, it is desirable to introduce at an early stage the integral as the limit of a sum. The writer is accustomed to spend about fifteen lectures of the first half of the first course in calculus on this subject and to have a great many problems, not merely in volumes, centers of gravity and moments of inertia, but in fluid pressures, attractions, and work and energy worked out by the students.‡ His experience has been that this topic is an

* "Lehrbuch der Differential- und Integralrechnung," 4th ed., Weimar, 1895.

† "Kurzgefasstes Lehrbuch der Differential- und Integralrechnung," on which is based Young and Linebarger's "The Elements of the Differential and Integral Calculus," New York, 1900.

‡ A good collection of problems of the class which the writer has in mind is found in Professor Byerly's "Problems in Differential Calculus," Boston, Ginn & Co. This collection also includes problems in the integral calculus.

especially instructive one. Right ideas regarding the fundamental conceptions of the calculus and a correct understanding of its principles are acquired, not through a lengthy exposition at the outset when the mind of the student is unprepared for what he is to receive, but by the teacher's constantly training him all through the course to form right habits of thought in these matters. A valuable aid in this direction may be found in these introductory chapters of Professor Gibson's book, when used for collateral reading, and in many passages of the later chapters in which the author warns the student against a false conception or trains him to use the rigorous methods of modern analysis.

Chapter IX begins with Rolle's theorem and the theorems of mean value. The whole modern treatment of the differential and integral calculus rests to a large degree on these important theorems and it is indicative of the modern tendencies of the work before us that these theorems not only are given a leading position, but also are applied systematically throughout the remainder of the book. Thus the error made in the older treatises on the calculus in the proof of the theorem that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ is pointed out in Chapter XI and it is shown by means of a simple example that the assumption made in these proofs to the effect that

$$\lim_{h=0}\,[\lim_{k=0} f(h, k)] \quad \text{and} \quad \lim_{k=0}\,[\lim_{h=0} f(h, k)]$$

have the same value is unwarranted; for, if for example $f(h, k) = (h + 2k)/(h + k)$, the first of these double limits has the value 1 and the second the value 2. Then follows a rigorous proof of the theorem.*

Chapter X is entitled: Derived and Integral Curves, Integral Function. Derivatives of Arcs and Volume of a Surface of Revolution, Polar Formulæ, Infinitesimals.

Chapter XI, Partial Differentiation, contains a treatment

* This is the proof that is now commonly given in continental textbooks on the calculus. It is often ascribed to Schwarz; but it appears in substance in Serret's Cours de calcul différentiel und intégral, 1st ed., 1868, pp. 76–78, where the statement is made that it is due to Ossian Bonnet. Schwarz refers to this passage in his memoir. In Bonnet's proof the continuity of both the derivatives of the second order which enter is employed, and it is necessary only to incorporate the requirement of the continuity of these derivatives into the statement of the theorem to make Bonnet's proof, which is based on the theorems of mean value, rigorous. What Schwarz actually did was to show (*a*) that the theorem as stated by Serret without any requirement regarding the continuity of the two derivatives of the second order is false; (*b*) that it is enough to require the continuity of *one* of these derivatives.

of this subject which in many respects is admirable. In the theory of partial derivatives and differentials two theorems are fundamental.

Theorem A. If u is a function of the independent variables $x, y, \cdots$ and if each of these is in turn a function of the independent variables $r, s, \cdots$, then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \cdots$$

with similar equations for $\partial u/\partial s$, etc., provided that the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\cdots$ on the one hand, and on the other the partial derivatives $\frac{\partial x}{\partial r}$, $\frac{\partial y}{\partial r}$, $\cdots$, $\frac{\partial x}{\partial s}$, $\frac{\partial y}{\partial s}$, $\cdots$ are continuous.

Theorem B. If the differential of u, du, is defined by the formula*

$$du = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \cdots$$

when $x, y, \cdots$ are the independent variables, then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \cdots,$$

no matter whether $x, y, \cdots$ are the independent variables or whether $x, y, \cdots$ are functions of $r, s, \cdots$ and these latter are regarded as the independent variables, provided merely that the first partial derivatives that enter are continuous.

The second of these theorems is indispensable in establishing the legitimacy of the ordinary definition of du, both for functions of several variables and for those of a single variable, a fact generally overlooked by English writers on the calculus, not excepting the present author.

In presenting the subject to beginners it is well to commence with the case that the number of variables in the group $r, s, \cdots$ is one,—call the variable t,—and then the formula of Theorem A becomes

* The author denotes increments Δx, Δy, Δu, etc., throughout the book by δx, δy, δu, etc. This notation is unfortunate inasmuch as it is used by mathematical writers for things not increments much oftener than is the large Δ, and the student is likely sooner or later to get confused ideas as to what meaning is to be attached to δu.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \cdots.$$

This Professor Gibson does and his presentation is clear and accurate. But in the general case Theorem *B* is entirely omitted and only a special case of Theorem *A* appears in the course of an example that is worked out in the text. The few pages still needed for a fairly complete treatment of the subject of partial differentiation would have been a welcome addition to the chapter before us.

Chapter XII, Applications to the Theory of Equations, will command the attention of all who are engaged in teaching calculus, mechanics, or any of the other physical sciences to which calculus is applied. Newton's method of approximating to the roots of an equation is set forth, the steps are illustrated graphically and a check for the error is obtained. One of the equations, for example, to the solution of which the method is applied is $x + \sin x - \frac{1}{2}\pi = 0$. The method of successive approximations follows. The expansion of a root of the equation $x = \varphi(x)$ into a series is taken up and applied to finding the roots of the equation $x = \tan x$. Numerous examples are given for the student to work out; these include, in particular, the determination of the constant a of the catenary from the equation

$$l = a(e^{\frac{c}{2a}} - e^{-\frac{c}{2a}}),$$

l and c being given and l being only slightly greater than c; *e. g.*, $l = 105$, $c = 100$. There is an excellent paragraph on Proportional Parts, with a simple check for the error. The chapter closes with a paragraph on Small Corrections.

In Chapter XIII, pages 262–297, the formal methods of integration are treated at some length. In connection with this chapter the student should learn to use some good table of integrals.*

Chapter XIV, pages 298–323, deals with properties of definite integrals. The theorems of mean value are given and application of the first law is made to the approximate computation of certain definite integrals, one of the examples being to show that, if $n > 2$,

$$.5 < \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^n}} < .524.$$

* Cf., for example, B. O. Peirce: A Short Table of Integrals, Ginn & Co., Boston, which in the revised edition of 1899 contains 569 formulas of integration and in addition many other useful formulas and tables.

The chapter closes with an account of Amsler's planimeter.

It is not until Chapter XV, pages 324–351, that the integral as the limit of a sum is introduced. The definite integral was defined (page 262) as the difference between two values of the indefinite integral

$$\int_a^b f(x)dx = [D_x^{-1} f(x)]_a^b,$$

and the existence of an indefinite integral was shown by reference to the area under the curve $y = f(x)$. While this is not the customary definition of a definite integral, the discrepancy so far as the student is concerned is less startling than would appear at first sight, since the foundation on which he builds his integral calculus is, with Professor Gibson as with others, the area under the curve $y = f(x)$. Nevertheless, of the problems of the integral calculus which can be treated either by determining a function u so that it will satisfy the conditions

$$\frac{du}{dx} = f(x), \qquad u|_{x=x_0} = 0,$$

or by means of the limit of a sum, it seems to us undesirable to solve very many by the former method. For, the latter method is of fundamental importance in the calculus; it is instructive, and it is not beyond the powers of the beginner at an early stage. Moreover, it is the only method that there is for the problems of the calculus that lead to multiple integrals and it is precisely the training that the simple integrals afford which the student needs in order to deal readily with these problems. The chapter gives methods for the approximate computation of definite integrals (the Trapezoidal Rule, Simpson's Rule). It closes with double integrals

Chapter XVI, pages 352–374, is entitled Curvature, Evolutes, and includes a treatment of the cycloid.

In Chapter XVII, Infinite Series, the text of which occupies but twelve pages, the usual tests for convergence are obtained, the reasoning on which they rest being set forth with admirable clearness. The ordinary sufficient condition that a series of continuous functions should yield a continuous function is established and applied to power series. We note with satisfaction a rigorous proof on page 388 that, if a power series vanishes for all values of its argument within a certain interval, each of its coefficients must vanish. The chapter, though somewhat brief, is well

done, and its presence is characteristic of the modern tendencies of Professor Gibson's work.*

Chapter XVIII, pages 390–407, contains an excellent presentation of Taylor's Theorem and concludes with a good collection of examples. A paragraph is devoted to the differentiation and the integration of series. In the statement of the tests (Theorems I and II, pages 399, 400) the condition that the terms of the series be *continuous* functions should be included.†

A chapter on Taylor's Theorem for a Function of Two or More Variables; Applications, and one on Differential Equations conclude the book. In the former chapter is contained a rigorous treatment of Indeterminate Forms such as is now usual in continental treatises on the calculus. The latter is, in the language of the preface, "designed to illustrate the types of equations most frequently met with in dynamics, physics, and mechanical and electrical engineering."

In the foregoing account and critique of the book before us the reviewer is conscious of having laid special stress on the author's development of the fundamental principles of the calculus and he fears that some readers of the BULLETIN may have received the impression that Professor Gibson's work is an imitation of certain continental treatises on the calculus which, while containing much that is of interest to the specialist in the theory of functions of a real variable, appear to the general student to be difficult and arid. This is, however, by no means the case. The style is simple and the presentation is such as to appeal to the student who has appreciation for the exact sciences. The choice of material is exceptionally good and covers a wide range. The problems appended to each chapter for the student to solve are numerous and well selected.

* On p. 382 near the top: "The series (the exponential series, §49) is therefore convergent for every finite (positive) value of x." The restriction *finite* is incorrect, for x is a *constant* with reference to the convergence of the series and there is no such thing in the calculus as an *infinite constant*. One of the excellent features of Professor Gibson's book is that the author repeatedly emphasizes this fact; thus, on p. 195, he insists that an infinitesimal is not a constant, but is a *variable*. Clearly, the word "finite" was unintentional, but it is an oversight that may easily lead the beginner into error.

† On p. 400, near the middle: "We will now show that the series obtained by differentiating the power series is uniformly convergent when x is *within* the interval of convergence * * *." This is wrong as it stands; the series is *not* in general uniformly convergent for this range of values for x. It may, however, be corrected by replacing "x" by the words "the interval for x: $a \leqq x \leqq b$."

While many teachers may not see their way to using the book as a text during the early part of the course, it will nevertheless commend itself for collateral reading from the very beginning, and its ultimate introduction as the sole or chief text-book of the latter part of the first course and of parts of the second course is a matter that teachers of calculus will do well carefully to consider.

W. F. OSGOOD.

HARVARD UNIVERSITY, CAMBRIDGE, MASS.,
January, 1902.

SHORTER NOTICES.

Éléments de la Théorie des Nombres. Par E. CAHEN, Professeur de Mathématiques spéciales au Collège Rollin. Paris, Gauthier-Villars, 1900. 8vo, viii + 403 pp. Price, 12 fr.

THE purpose of the author, as stated in his preface, is to supply the lack of a modern treatise in French on the theory of numbers. The works of Bachmann, Dirichlet-Dedekind, and Tchébyscheff are mentioned; but no reference is made to French books. But the second volume of Serret's Algèbre Supérieure treats of congruences, quadratic residues, Galois imaginaries, and the number of primes in a given interval. Of these topics, the latter two are not treated by Cahen. Again, chapters IV, V and XIV of Tannery's Leçons d'Arithmétique form a most excellent introduction to the ordinary theory of numbers, aside from quadratic forms.

Cahen treats not merely of the properties of integers, congruences and quadratic forms, but also of irrational numbers, particularly of algebraic numbers of the second degree. Fractions are introduced (on page 20) from the standpoint of pure analysis as symbols each defined by a system of two integers, called numerator and denominator. By definition

$$\frac{a}{b} \lessgtr \frac{c}{d} \quad \text{according as} \quad ad \lessgtr bc.$$

The usual results for the sum and product of two fractions are taken as the definitions of sum and product.

Irrational numbers and operators upon them are defined by means of Dedekind's *cut*.

In treating congruences of general degree with respect to a prime modulus (pages 69–77), the author leaves to the reader the proof of eight theorems on the divisibility of polynomials (mod p), analogous to theorems previously established for integers. In this matter the German texts are obviously simpler.

Two proofs are given of the law of reciprocity of quadratic residues, that by Pastor Zeller, and one of those due to Kronecker. The latter proof is said to be much more artificial than the former, at least at first view. This is not apparent, as both depend immediately upon the celebrated lemma of Gauss. This lemma is given on page 118 without mention of Gauss. In general, references are given sparingly and usually several pages after the theorems are stated.

The results on pages 131–133 are stated carelessly, the limitation "premiers" being omitted in four places. The results stated hold only for prime numbers.

Cahen employs (page 136) Tannery's modification of the Schering-Kronecker method of defining Jacobi's symbol $\left(\frac{n}{P}\right)$, P being any odd number, to equal $(-1)^{\mu}$, where μ is the number of negative minimum residues (mod P) of the products $n, 2n, \cdots, \frac{1}{2}(P-1)n$.

Chapter VI (pages 201–315) is devoted to binary quadratic forms and their application in the solution of an indeterminate equation of the second degree in two variables. For the form

$$f \equiv ax^2 + 2bxy + cy^2,$$

the quantity $b^2 - ac$ was called by Gauss the *determinant* of the form f, and designated by the letter D. This notation has since been employed by Bachmann, Dirichlet, Dedekind, etc. But Cahen makes its negative, the *discriminant* $ac - b^2$, play the fundamental rôle, employing for it the same designation D.

In adopting (page 208) the notation ABC for the product $(AB)C$, where A, B, C are binary linear substitutions, Cahen fails to state that $(AB)C = A(BC)$, a result very simply proved.

With Gauss and Dirichlet the form f of negative determinant is called *reduced* when $c \geqq a \geqq 2\,|b|$. Cahen adds the conditions $2b \neq -a$; $b \geqq 0$ if $a = c$, thereby ruling out one of the forms in each of two pairs of equivalent Gauss-reduced forms. In Klein's geometric phraseology, these

added conditions indicate which boundaries are to form part of the fundamental region [Klein, Ausgewählte Kapitel der Zahlentheorie I, p. 216].

A simplification is made by Cahen by employing as roots of a form the reciprocals of the values used by Dirichlet. Then, in Cahen's notation, a substitution which transforms a form into a second transforms the roots of the second into the roots of the first form.

On page 301, sixth line from bottom, and on page 306, first line of §376, the word impropre should read propre.

The concluding pages (316–400) are devoted to notes and tables, the latter being borrowed from Tchébyscheff. There is a note on prime numbers in which are proved special cases of Dirichlet's theorem on an arithmetical progression. A note on the decomposition of numbers into prime factors shows how the problem can be solved by finding a quadratic form $x^2 + Dy^2$ which represents the given number, using the tables on pages 391–400. In the headlines to pages 397–400, $x^2 + \Delta y^2$ should read $x^2 - \Delta y^2$. There is a note on the calculation of primitive roots of prime numbers and tables (pages 375–390) giving the primitive roots and indices for all prime numbers < 200. The final note gives Gauss's theory of complex integers $a + bi$, their geometrical representation being emphasized.

Cahen's book will prove of special interest to those students who desire numerous illustrative examples and numerical applications of the general theorems. The amount of detail, which has added considerably to the size of the book, can not fail to allure the reader to the fascinations of number theory.

L. E. DICKSON.

Essays on the Theory of Numbers: I. *Continuity and Irrational Numbers.* II. *The Nature and Meaning of Numbers.* By RICHARD DEDEKIND. Authorized translation by W. W. Beman. Chicago, The Open Court Publishing Company, 1901. 115 pages.

THE essays of Dedekind, Stetigkeit und irrationale Zahlen (Braunschweig, 1872), and Was sind und was sollen die Zahlen? (Braunschweig, 1888) have already become classics in the literature of mathematics. In giving a fairly literal translation of them, Professor Beman performs a service for which one must feel grateful, especially as one needs whatever advantage one's own language gives in attempting to master the abstruse second essay.

The word Abbildung is translated transformation (page

50). Then one has a transformation of an element s, or system S, *into* an element $\varphi(s)$, or a system $\varphi(S)$, and also (page 56) a transformation of S *in* Z. [The latter denotes a transformation of S into $\varphi(S)$, a part of Z.] This use of the word tran-formation is not easily associated with its ordinary use. The translation "representation" for Abbildung has been used by others. By its adoption, the language becomes very smooth. Then, under the representation φ, a system S is represented by $\varphi(S)$. If the latter is a part of a system Z, we have a representation of S in Z.

In rendering more accessible various important foreign books, the Open Court Company is doing excellent work for science.

L. E. Dickson.

Annuaire pour l'An 1902, publié par le Bureau des Longitudes. Paris, Gauthier-Villars.

The Annuaire for the year 1902 does not differ materially from its predecessors. The information on the various subjects which it treats is, as usual, brought up to date; otherwise the body of the volume contains no material changes. The only matters which call for special remark are the notices placed at the end of the volume. The committee which has charge of the Annuaire always chooses subjects for these notices which shall be scientific and of general interest. The current Annuaire is no exception to the rule.

M. Poincaré writes a luminous article of thirty-four pages, on wireless telegraphy. He gives a simple account of the main principles which underlie this latest development in the applications of electricity. As usual, he carries the reader by easy stages up to the most recent results, including those of Marconi, so far as they are known.

M. Cornu develops, with much detail, the theory and practice of polyphase currents. Some parts of this article will perhaps be a little tedious to the mathematician, as several pages, here and there, are devoted to elementary explanations of harmonic motion, and the composition of harmonic motions of the same phase and different amplitudes; but this is doubtless an advantage for many of the readers whom this volume reaches. Considerable space is devoted to the construction of the most modern forms of dynamos.

In the third appendix M Guyou makes a plea for the use of the decimal division of the angle. He admits that it is hopeless to ask astronomers to change from the degree to

the grade on account of the large masses of results which have been published in degree measure and the trouble of continual comparison between the two systems. He thinks, that for navigational and geodetic purposes, the decimal division is far easier to handle, and that the trouble required to make the change is not prohibitive.

Finally, M. Janssen gives his annual report of the work done in the Observatory at the summit of Mont Blanc. This time, he adds to it the history of the foundation of the observatory. It is interesting to notice that the lowest temperature hitherto recorded by the self-registering instruments (−45° C.) was obtained last year.

ERNEST W. BROWN.

NOTES.

THE spring meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago on Saturday, March 29.

THE library of the AMERICAN MATHEMATICAL SOCIETY is now bound and catalogued and ready for use. Members desiring to draw books may do so on applying to the Librarian. Books will be sent, express due, and must be returned within four weeks, express prepaid. Letters of inquiry concerning books, publishers, prices, etc., will be answered by the Librarian, so far as the facilities of the Society's library or of the library of Columbia University permit. It is the desire of the Council and of the Librarian to make the library as helpful as possible to the members. Circulars have been sent to learned societies and institutions, to editors of mathematical journals, and to publishers, asking for coöperation in the plans for enlarging the collection.

It would be fortunate if this library could become an exponent of the mathematical work of the Society. To this end members are invited to contribute copies of their publications, whether of books or of monographs, and to continue to do this as new works appear. Such a library would soon, with slight sacrifice on the part of each member, become an invaluable record of the advance of mathematical thought in America. The Librarian will be glad to acknowledge the receipt of any works of this nature, to inscribe the information that they are gifts from the author to the Society, and to state this fact in the catalogues as they

appear. If members will give this matter immediate attention the library will begin substantial growth at once.

THE preliminary programme of the meeting of the London mathematical society, to be held February thirteenth, announced papers "On the density of linear sets of points," by Mr. W. H. YOUNG, and "On plane cubics," by Professor A. C. DIXON.

AN international congress of the historical sciences is to be held in Rome in April of this year. Professor V. CERRUTI, of the University of Rome, is president of the section devoted to the mathematical and natural sciences, and Professor A. FAVARO, of the University of Padua, of the subsection on mathematics and physics.

THE French ministry of public instruction has recently issued the minutes of the Paris congress of mathematicians, drawn up by the general secretary, Professor E. DUPORCQ. The brochure contains in seventeen pages a brief summary of the scientific and business proceedings; but a list of those participating in the Congress is not included.

THE proceedings of the Berlin mathematical society are to be published in the *Archiv der Mathematik und Physik*. The *Jahresbericht* of the German mathematical society reports the titles and authors of papers read before many mathematical associations; in particular it contains brief abstracts of the proceedings of the Göttingen mathematical society. The *Jahresbericht* also publishes the tables of contents of the current numbers of mathematical journals.

A COURSE of six public lectures will be delivered by Dr. ALEXANDER MACFARLANE at Lehigh University, March 14–25, on the life and work of the following British mathematicians: Maxwell, H. J. S. Smith, Rankine, Sylvester, Tait and Kelvin.

CORNELL UNIVERSITY.—The announcement for the summer session, July 7 to August 16, includes the following advanced courses in mathematics:—By Professor L. A. WAIT: Differential calculus, two hours.—By Dr. J. I. HUTCHINSON: Integral calculus, five hours; History of mathematics, five hours.—By Dr. H. F. STECKER: Differential equations, five hours; Non-euclidean geometry, three hours; Theory of functions, three hours.—By Dr. W. B. FITE: Introduction to the theory of groups, three hours.

UNIVERSITY OF MUNICH.—The following courses in mathematics are announced for the summer semester of 1902:—By Professor F. LINDEMANN: Integral calculus; Theory of substitutions and of higher algebraic equations; Mechanics of deformable bodies; Mathematical seminary.—By Professor A. PRINGSHEIM: Selected chapters from the theory of algebraic functions; Fourier series.—By Dr. H. BRUNN: Elements of higher mathematics for students of all faculties.—By Dr. K. DÖHLEMANN: Descriptive geometry, with exercises; Selected chapters from modern geometry.—By Dr. E. VON WEBER: Solid analytical geometry with exercises; Theory and applications of determinants.

UNIVERSITY OF PARIS.—The following mathematical courses are announced by the faculty of sciences for the semester opening March 1, 1902, each course consisting of two lectures per week:—By Professor E. PICARD: Functions of several variables, especially the algebraic functions of two variables and the associated transcendentals.—By Professor E. GOURSAT: Differential equations continued.—By Professor P. APPELL: General laws of motion of systems, analytic mechanics, hydrostatics and hydrodynamics.—By Professor J. BOUSSINESQ: Theory of light.—By Professor G. KOENIGS: Theory of elasticity.—By Professor L. RAFFY: Integral calculus and its applications to mechanics and physics.

Conferences are conducted by Professor RAFFY in the integral calculus, Professor J. HADAMARD in the calculus and in higher analysis, by Professor P. PUISEUX in mechanics and astronomy, and by Professor H. ANDOYER, Professor HADAMARD, and Dr. E. BLUTEL preparatory to the agrégation.

UNIVERSITY OF TÜBINGEN.—The following mathematical courses are announced for the summer semester of 1902:—By Professor H. STAHL: Elementary analysis, three hours; Theory of functions, three hours; Seminar, one hour.—By Professor A. VON BRILL: Analytical mechanics, five hours; Selected chapters from the theory of curves of double curvature, two hours; Seminar, one hour.—By Professor L. MAURER: Higher analysis, four hours; Theory of invariants of binary forms, two hours; Exercises in higher analysis, two hours.

THE Belgian academy of sciences proposes the following prize question for 1902:

A contribution to the algebraic and geometric theory of forms in $n > 3$ variables.

Papers may be written in French or Flemish, and should be sent under the usual conditions as to anonymity to the permanent secretary of the Academy at Brussels. The value of the prize is 600 francs.

THE Amsterdam scientific society has recently issued its annual list of prize questions. Fifteen subjects in mathematics and physics are proposed. Copies of the announcement may be obtained from the secretary of the society, Professor D. J. KORTEWEG, Vondelstraat, 104, Amsterdam.

THE second number of the third volume of *Bibliotheca Mathematica*, which appeared recently, contains an elaborate account of the mathematical work of the late Professor E. Beltrami. The article is written by Professor G. LORIA, of the University of Genoa, and is illustrated by a portrait of Beltrami.

THE *Archiv der Mathematik und Physik* contains in each number a Sprechsaal for the Encyclopædia of the mathematical sciences, to which it invites contributions in the way of additions and corrections.

PRELIMINARY to the publication of the works of Leibniz, the Berlin academy of sciences has entrusted to Dr. W. KABITZ and Dr. P. RITTER the preparation of a complete list of Leibniz's writings.

BEGINNING with the year 1905, the Ecole polytechnique at Paris will discontinue the use of sexagesimal trigonometric tables, replacing them by centesimal tables with five places which are held to suffice completely for the daily use of the officers and engineers.

A NEW associate professorship of mathematics is to be established at the Vienna technical high-school.

THE University of Strassburg conferred four doctorates in mathematics, out of a total of fourteen in all subjects, during the summer semester of 1901.

PROFESSOR G. PEANO, of the University of Turin, assisted by a number of Italian mathematicians, has in preparation a dictionary of mathematics.

AT the celebration of the twenty-fifth anniversary of the founding of the Johns Hopkins University, the honorary degree of doctor of laws was conferred upon Professor SIMON NEWCOMB.

ON the fourteenth of December a formal farewell reception was held at the Ecole polytechnique in honor of Pro-

fessor A. MANNHEIM upon his retirement from the chair of geometry, which he has filled since 1864, and which he now relinquishes under the age limit. The reception, which was held in the large amphitheater of the school, was arranged by a representative committee and was attended by the minister of war and many prominent scientists and officials. Several addresses were made, M. E. Rouché and the minister of war being among the speakers. The former dwelt particularly upon M. Mannheim's contributions to science in his Principes et développements de la géométrie cinématique. The occasion was further marked by the presentation to M. Mannheim of a token of remembrance in the form of a bronze by Coutan.

PROFESSOR G. MITTAG-LEFFLER has recently been elected foreign associate of the Royal society of Naples and associate of the Royal academy of Belgium.

MR. T. J. I'A BROMWICH, fellow of St. John's College, Cambridge, has been appointed professor of mathematics in Queen's College, Galway, in succession to Professor A. C. DIXON, now professor of mathematics in Queen's College, Belfast.

MR. R. J. PARANJPYE, senior wrangler at Cambridge University three years ago, and recently fellow of St. John's College, has returned to his native country to accept a professorship at Fergusson College, Poona.

MR. FREDERICK PURSER has been elected to the chair of natural philosophy in Trinity College, Dublin, lately vacated by Dr. TARLETON.

SEVERAL scientific societies will unite in tendering a reception to Lord KELVIN at Columbia University on the evening of April 21.

DR. C. M. GULDBERG, professor of mathematics in the technical school at Christiania, Norway, died on January 14.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BASS (E. W.). Elements of differential calculus. 2d edition revised. New York, Wiley, 1902. 12mo. 8+356 pp. Cloth. $4.00

BURKHARDT (H.). See ENCYKLOPÄDIE.

DEDEKIND (R.). Ueber die Permutationen des Körpers aller algebraischen Zahlen. (Festschrift zur Feier des 150jährigen Bestehens der k. Gesellschaft der Wissenschaften zu Göttingen.) Berlin, Weidmann, 1901. 4to. 17 pp. M. 2.00

DICKSON (L. E.). Representation of linear groups as transitive substitution groups. 1901. 4to. (*American Journal of Mathematics* 23, pp. 337–377.)

——. The configurations of the 27 lines on a cubic surface and the 28 bitangents to a quartic curve. 1901. 8vo. (*Bulletin of the American Mathematical Society*, 2d series, Vol. 8, pp. 63–70.)

——. The groups of Steiner in problems of contact. 1902. 4to. (*Transactions of the American Mathematical Society* 3, pp. 38–45.)

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Herausgegeben im Auftrage der Akademien der Wissenschaften zu München und Wien und der Gesellschaft der Wissenschaften zu Göttingen, sowie unter Mitwirkung zahlreicher Fachgenossen. (In 7 Bänden.) Vol. II: Analysis, in 2 Teilen, redigiert von H. Burkhardt. Teil 2, Heft 1: W. F. Osgood, Allgemeine Theorie der analytischen Funktionen (*a*) einer und (*b*) mehrerer komplexen Grössen; W. Wirtinger, Algebraische Funktionen und ihre Integrale. Leipzig, Teubner, 1901. 8vo. 175 pp. M. 5.20

GUNDERSEN (C.). On the content or measure of assemblages of points. Submitted in partial fulfillment of the requirements for the degree of doctor of philosophy, in the Faculty of pure science, Columbia University. New York, 1901. 4to. 24 pp.

HAMEL (G.). Ueber die Geometrien, in denen die Geraden die Kürzesten sind. (Diss.) Göttingen, 1901. 8vo. 90 pp.

Χατζιδάκι (I. H.). Εἰσαγωγὴ εἰς τὴν ἀνωτέραν ἀλγεβραν. 2d edition. Athens, 1898.

HILBERT (D.). Ueber das Dirichlet'sche Prinzip. (Festschrift zur Feier des 150jährigen Bestehens der k. Gesellschaft der Wissenschaften zu Göttingen.) Berlin, Weidmann, 1901. 4to. 27 pp. M. 2.00

KORTEWEG (D. J.). Oplossingen van vraagstukken uit de analytische meetkunde van het platte vlak en der ruimte, voorkomende in Briot et Bouquet, Leçons de géométrie analytique, benevens een gelijk aantal vraagstukken ter oefening. 3ter druk. 's Hertogenbosch, 1901. 8vo. 8+192 pp. M. 5.50.

LEONARDI (G.). Sull' omografia della specie *n*. Acireale, Tipografia dell' *Etna*, 1901. 8vo. 32 pp.

OSGOOD (W. F.). See ENCYKLOPÄDIE.

SCHÜTTENHELM (A.). Ueber eine besondere Art Cremona'scher Transformationen. (Diss.) Strassburg, 1901. 8vo. 47 pp.

TORRELLI (G.). Sulla totalità dei numeri primi fino ad un limite assegnato; monografia, premiata dalla r. Accademia delle scienze fisiche e matematiche di Napoli nell'adunanza del 15 dicembre 1900. Napoli, Tipografia dell'Accademia, 1901. 4to. 215 pp., 1 table. (Memoria estratta dal Vol. XI, serie 2a, No. 1, degli *Atti della r. Accademia delle scienze fisiche e matematiche di Napoli.*)

VAN VLECK (E. B.). A determination of the number of real and imaginary roots of the hypergeometrric series. 1902. 4to. (*Transactions of the American Mathematical Society* 3, pp. 110–131.)

WILCZYNSKI (E. J.). Geometry of a simultaneous system of two linear homogeneous differential equations of the second order. 1901. 4to. (*Transactions of the American Mathematical Society* 2, pp. 343–362.)

——. Reciprocal systems of linear differential equations. 1902. 4to. (*Transactions of the American Mathematical Society* 3, pp. 60–70.)

WILHELM (J.). Die Kegelschnitte mit einem gemeinschaftlichen Brennpunkt in ihrem Zusammenhange mit den Kreisen der Ebene. (Diss.) Strassburg, 1901. 8vo. 40 pp.

WIRTINGER (W.). See ENCYKLOPÄDIE.

ZOLL (O.). Ueber Flächen mit Schaaren von geschlossenen geodätischen Linien. (Diss.) Göttingen, 1901. 8vo. 45 pp.

II. ELEMENTARY MATHEMATICS.

AMES (A. F.). See MCLELLAN (J. A.).

BEMAN (W. W.). See SUNDARA ROW (T.).

BOUVART (C.) et RATINET (A.). Nouvelles tables de logarithmes à cinq décimales (division centésimale, conforme à l'arrêté ministériel du 3 août 1901), à l'usage des candidats aux écoles polytechnique et Saint-Cyr, contenant: 1° les logarithmes des nombres entiers, de 1 à 10,000; 2° une table à sept décimales pour les calculs relatifs aux intérêts composés; 3° un tableau permettant de calculer rapidement la longueur d'un arc en fonction du rayon; 4° une table pour la conversion des divisions sexagésimales en divisions centésimales; 5° une table pour la conversion des divisions centésimales en divisions sexagésimales; 6° une table trigonométrique centésimale. Paris, Hachette, 1902. 8vo. 128 pp. Fr. 2.00

CAVEZZALI (A.). Esercizi e nozioni di aritmetica e geometria, compilati in conformità dei programmi governativi 29 novembre 1894 e distribuiti con metodo ciclico. Parte III, per la classe 16mo. 120 pp. Fr. 0.40

CIAMBERLINI (C.). Esercitazioni e ricrezioni geometriche ad uso degli alunni delle scuole elementari. Lanciano, Carabba, 1902. 16mo. 45 pp. Fr. 0.30

DEEL (G. A.). Practical rapid calculator for the use of schools where the commercial branches are taught, and self-instruction; embracing the shortest and most practical methods of calculating for a large variety of trades and professions. Poughkeepsie, N. Y., Deel, 1901. 76 pp. Cloth. $0.50

DEWEY (J.). See MCLELLAN (J. A.).

DICKSON (L. E.). College algebra. New York, Wiley, 1902. 12mo. 7+214 pp. Cloth. $1.50

DURET (G.). See SURIER (A.).

GAZZANIGA (P.). See VERONESE (G.).

GRASSI (F.). See SERRET (G. A.).

HALL (H. S.). Algebraical examples: supplementary to Hall and Knight's algebra for beginners and elementary algebra (chapters 1 to 27). London and New York, Macmillan, 1901. 12mo. 8+172 pp. Cloth. $0.50

KELLOGG (A. M.). Elements of algebra. New York, Kellogg and Co., 1901. 12mo. 47 pp. Cloth. $0.25

KINKELIN (D. H.). See RIEM (J.).

LEATHEM (J. G.). See TODHUNTER (I.).

MCLELLAN (J. A.)., DEWEY (J.)., and AMES (A. F.). Public school arithmetic for grammar grades, based on McLellan and Dewey's "Psychology of number." New York, Macmillan, 1902. 12mo. 12+369 pp. $0.60

MILNE (W. J.). A key to the "Standard arithmetic" and the "Mental arithmetic." New York, American Book Co. [1901]. 74 pp. Cloth. $1.00

MORE (E.). The trisection of a given angle geometrically solved and illustrated. Grand Rapids, Mich., Gregory Wickes Pub. Co., 1901. 7 pp. $1.00

MÜLLER (H.). Die Elementar-Planimetrie. Ein methodisches Lehrbuch. Leipzig, Teubner, 1901. 8vo. 8+188 pp. Boards. M. 2.40

RATINET (A.). See BOUVART (C.).

RIEM (J.). Tables de multiplication pour le commerce et l'industrie. Avec un avant-propos par D. H. Kinkelin. 2e édition. Paris, 1901. 4to. Fr. 8.50

SABRÁZ Y CAUSAPÉ (T.). See SANCHEZ RAMOS (E.).

SÁNCHEZ RAMOS (E.) Y SABRÁS Y CAUSAPÉ (T.). Segundo curso de matemáticas; nociones de geometría. 2a edición. Vitoria, Egaña, 1901. 8vo. 124 pp.

SERRET (G. A.). Trattato di trigonometria piana e sferica tradotta in italiano sulla settima edizione francese, col consenso dell'autore, da F. Grassi. 5a edizione, can note ed aggiunte del traduttore, 1,000 esercizi colla soluzione, coll'aggiunta di un formulario di matematica e fisica. Torino, Bocca, 1902. 16mo. 312 pp. Fr. 3.00

SMITH (D. E.). See SUNDARA ROW (T.).

SPERINDEO (G.). Appunti di geometria per le prime tre classi ginnasiali. 2a edizione. Napoli, D'Auria, 1901. 16mo. 90 pp. Fr. 0.80

SUNDARA ROW (T.). Geometric exercises in paper folding; edited and revised by W. W. Beman and D. E. Smith. Chicago, The Open Court Publishing Co., 1901. 12mo. 14+148 pp. Cloth.

SURIER (A.) et DURET (G.). L'arithmétique simplifiée, en concordance avec la géométrie et le système métrique (Cours moyen et supérieur.) Mille cinq cent cinquante problèmes. Paris, Delalain, 1901. 8vo. 8+272 pp. Fr. 1.80

TODHUNTER (I.). Spherical trigonometry. For colleges and schools. Revised by J. G. Leathem. London, Macmillan, 1902. 12mo. 288 pp. Cloth.

VERONESE (G.). Elementi di geometria ad uso dei ginnasi e licei e istituti tecnici (1° biennio), trattati con la collaborazione di P. Gazzaniga. Parti I–II. 2a edizione. Verona e Padova, Drucker, 1901. 8vo. Vol. I: 20+111 pp.; Vol. II: 4+212 pp. Fr. 4.50

WIMMENAUER (F.). Arithmetische Aufgaben nebst Lehrsätzen und Erläuterungen. 2te Ausgabe. Breslau, Hirt, 1901. 8vo. 7+312 pp. M. 3.25

III. APPLIED MATHEMATICS.

APPELL (P.). Traité de mécanique rationnelle. (Cours de mécanique de la faculté des sciences.) Vol. III: Equilibre et mouvement des milieux continus. Fasc. 2. Paris, Gauthier-Villars, 1900. 8vo. Pp. 225–456.

BAUSCHINGER (J.). Tafeln zur theoretischen Astronomie. Leipzig, Engelmann, 1901. 4to. 4+148 pp., 2 plates. Cloth. M. 12.00

BECKER (A.). Interferenzröhren für elektrische Wellen. (Diss.) Heidelberg, 1901. 8vo. 62 pp., 3 plates.

BRACE (D. B.). Laws of radiation and absorption: memoirs by Prévost, Stewart, Kirchhoff, and Kirchhoff and Bunsen. Contents: On the equilibrium of heat, by Pierre Prévost; Treatise on radiant heat (selections), by Pierre Prévost; Biographical sketch of Prévost; account of some experiments on radiant heat, by Balfour Stewart; Biographical sketch of Stewart; On the relation between the emissive and the absorptive power of bodies for heat and light, by G. R. Kirchhoff; Biographical sketch of Kirchhoff; chemical analysis by spectral observations, by G. R. Kirchhoff and R. Bunsen; Biographical sketch of Bunsen; Bibliography. New York, American Book Co. [1901]. 7+131 pp. Cloth. $1.00

BUNSEN (R.) See BRUCE (D. B.).

CONCINA (U.). See ENRIQUES (F.).

DELASTELLE (F.). Mathématiques appliquées. Traité élémentaire de cryptographie. Paris, Gauthier-Villars, 1902. 8vo. 164 pp. Fr. 5.00

DU BOIS (A. J.). Mechanics of engineering. Volume I: Kinematics, statics, kinetics, statics of rigid bodies and of elastic solids. New York, Wiley, 1901. 4to. 634 pp. Cloth. $7.50

DUNLAP (H. C.) and JACKSON (C. S.). Slide rule notes. London, Simpkin, 1902. 4to. 68 pp. 3s.

ENRIQUES (F.). Lezioni di geometria descrittiva, pubblicate per cura di U. Concina. Bologna, 1901. 8vo. 121 pp., 24 plates. Fr. 12.00

HUYGENS (C.). Oeuvres complètes, publiées par la Société Hollandaise des sciences. Vol. IX: Correspondance 1685–90. La Haye, 1901. 4to. 662 pp., 1 portrait, 2 plates. M. 25.00

JACKSON (C. S.). See DUNLAP (H. C.).

JADANZA (N.). Elementi di geodesia. 5a edizione. Torino, Giorgis, 1902. 8vo. 542 pp.

KIRCHHOFF (G. R.). See BRACE (D. B.).

KITT (M.). Grundlinien der politischen Arithmetik. Zum Gebrauche an Handelsakademien, höheren Handelslehranstalten und zum Selbstunterrichte bearbeitet. Teil I: Zinseszins- und Rentenrechnung. Teil II: Tabellen. Leipzig, Teubner, 1901. 8vo. 4+78 pp. Cloth. M. 3.00

LANDRÉ (C. L.). See SCHOUTEN (P.).

PRÉVOST (P.). See BRUCE (D. B.).

SCHOUTEN (P.). Grondbeginselen der levensverzekeringswiskunde. Met een voorwort van C. L. Landré. Utrecht, 1901. 8vo. 8+152 pp. M. 3.80

STEWART (B.) See BRACE (D. B.).

WILDT (J.). Praktische Beispiele aus der darstellenden Geometrie für Lehranstalten mit bau- und kunstgewerblicher Richtung. Wien, Pichler, 1902. 4to. M. 17.00